Nonlinear relaxation between magnons and phonons in insulating ferromagnets

Valerij A. Shklovskij,[1] Viktoriia V. Mezinova,[1] and Oleksandr V. Dobrovolskiy[1,2,*]
[1]*Physics Department, V. Karazin Kharkiv National University, 61077 Kharkiv, Ukraine*
[2]*Physikalisches Institut, Goethe University, 60438 Frankfurt am Main, Germany*

(Received 14 June 2018; revised manuscript received 23 July 2018; published 4 September 2018)

Nonlinear relaxation between spin waves (magnons) and the crystal lattice (phonons) in an insulating ferromagnet is investigated theoretically. Magnons and phonons are described by equilibrium Bose-Einstein distributions with different temperatures, $T_s > T_l$. The magnon temperature is assumed to be much lower than the Debye temperature $T_s \ll \Theta_D$, which is justified at low temperatures. The nonlinear heat current from magnons to phonons is calculated *microscopically* in terms of the Cherenkov radiation of phonons by magnons. The results are discussed in comparison with the well-known theoretical results on nonlinear electron-phonon relaxation in metals [M. Kaganov, I. M. Lifshitz, and I. V. Tanatarov, J. Exp. Theor. Phys. **31**, 232 (1956)]. The elaborated theoretical description is relevant for spin-pumping experiments and thermoelectric devices in which the magnon temperature is essentially higher than the phonon one. The derived expression for the heat current can be used for calculation of the *nonlinear* heat boundary resistance in spin-caloritronic heterostructures.

DOI: 10.1103/PhysRevB.98.104405

I. INTRODUCTION

In recent years, spin caloritronics, which is concerned with the interplay between spin and heat currents in magnetic materials, has attracted great attention [1–3]. This attention is, in particular, motivated by recent discoveries related to thermal spin injection via the spin Seebeck effect [4–6], which can produce spin current densities that are two orders of magnitude larger than those produced via electronic or resonant excitation approaches. For instance, within the context of energy conversion applications, thermal spin transport provides conceptually new mechanisms for solid-state thermal-to-electrical energy conversion that may be used for waste heat recovery and temperature control [3]. Furthermore, the field of magnon spintronics has emerged [7], concerned with structures, devices and circuits that use spin currents carried by magnons, the quanta of spin waves. Analogous to conventional electric currents, magnon-based currents can be used to carry, transport and process information as an alternative to charge-current-driven spintronic devices [8,9]. Recently, pure magnonic spin currents in insulating ferromagnets featuring the absence of Joule heating and reduced spin-wave damping have been suggested for the implementation of efficient logic devices [10]. At the same time, spin waves can transport heat in the same manner that lattice excitations (phonons) transport heat through perturbations of the atom positions [11,12]. Heat transport by magnons and their relaxation on phonons become especially important in such insulating magnetic materials as, e.g., $Y_3Fe_5O_{12}$ [13], in contradistinction to metallic ferromagnets, whose thermal conductivity is dominated by the conduction electrons.

While the electron-phonon and magnon-phonon relaxation has been investigated in a series of theoretical works [11,14–20], the *nonlinear relaxation* of magnons on phonons—the subject of this work—has not been addressed theoretically so far. In this regard, the most closely related available theoretical work, which is similar in both the problem statement and the solution scheme, is the problem of nonlinear relaxation of electrons on phonons considered by Kaganov, Lifschitz, and Tanatarov (KLT) back in 1956 [15]. In that work, which is still the main model for analyzing experiments on the energy relaxation of excited electrons in metals [21–25], the nonlinear heat current Q from hot electrons at temperature T_e to cold phonons at temperature T_p in metals was calculated within the framework of the *two-temperature model*, with T_e and T_p being lower than the Debye temperature Θ_D. A nonlinear expression was obtained for the heat current $Q = A(T_e^5 - T_p^5)$ from electrons to phonons, where A is a constant expressed via the conductivity and the lattice parameters of the metal [15]. In particular, the KLT results have allowed for analyzing various aspects of the time-dependent dynamics of hot electrons in metallic thin films at low temperatures ($T \ll \Theta_D$) [18,20]. At the same time, while there have been phenomenological descriptions of spin relaxation beyond the linear regime relying upon the Fokker-Planck equation [26,27], so far the problem of relaxation between magnons and phonons in insulating ferromagnets has only been considered *microscopically* [28] in the *linear* regime $Q \sim (T_s - T_l)$, where T_s is the magnon temperature and T_l is the phonon temperature. At the same time, in experiments at low temperatures [29–31] and/or at high-power spin pumping [32], T_s can be essentially higher than T_l. Especially, this is justified in the case of strong spin currents; the development of approaches to their generation is currently a major direction in spintronics [33–35] and spin caloritronics [1–3]. In particular, while spin Seebeck measurements are usually carried out on bilayers (e.g., YIG and Pt), the large difference between T_s and T_l has an important implication for the heat current through their interface. Namely, the nonlinearity of the

*Corresponding author: dobrovolskiy@physik.uni-frankfurt.de

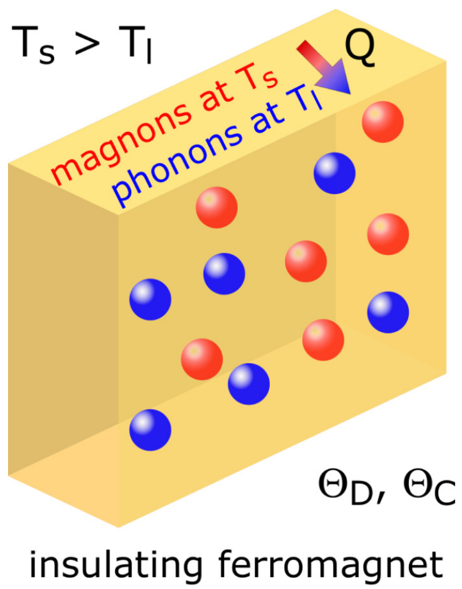

FIG. 1. Formulation of the problem: The nonlinear relaxation between magnons and phonons (denoted as red and blue spheres, respectively) is considered in an insulating ferromagnet. Θ_D, Debye temperature; Θ_C, Curie temperature. Magnons are characterized by the temperature T_s, which is essentially higher than the phonon temperature T_l but much lower than the Debye temperature Θ_D. The magnon subsystem is considered in the quasiequilibrium regime. The searched-for quantity is the nonlinear heat current Q from hot magnons to cold phonons.

thermal boundary (Kapitza) resistance [18,36] becomes most pronounced at low temperatures and it is expected to modify the linear-response calculation of the phonon, electron, and magnon temperature profiles applicable at room temperature [2]. Specifically, in the linear approximation T follows the Newton relation for the heat current $Q \approx (R^{-1})\Delta T$, where $\Delta T = T_s - T_l$ and $R(T)$ is an analog of the Kapitza resistance which evolves as $\backsim 1/T^3$. Obviously, $R(T)$ increases by three orders of magnitude as the temperature decreases from 300 to 30 K, while its increase reaches a factor of 10^6 in a low-temperature experiment at 3 K. It is therefore clear that even a not-too-large difference between T_s and T_l will strongly affect Q at low temperatures, thus necessitating a theoretical account for the nonlinear regime.

Here, we bridge this gap by considering the case of *nonlinear* relaxation between magnons and phonons when $T_s > T_l$ and derive expressions for the nonlinear heat current from magnons to phonons in an insulating ferromagnet. The problem is considered under the assumption that magnons are characterized by the temperature T_s, which is essentially higher than the phonon temperature T_l, but it is much lower than the Debye temperature Θ_D, which is justified at low temperatures [29–31].

II. MAIN RESULTS

Specifically, we consider the following problem. The nonlinear relaxation between spin waves (magnons) and the crystal lattice (phonons) is considered in an insulating ferromagnet (Fig. 1). In the ferromagnet, magnons are characterized by the temperature T_s, which is essentially higher than the phonon temperature T_l, i.e., $T_s > T_l$. The equilibration time for magnons in the ferromagnet is much smaller than the equilibration time between magnons and the crystal lattice [28,37,38]. This condition is justified at temperatures above 1 K [37]. Accordingly, the magnon subsystem is considered in the quasiequilibrium regime described by the conventional Bose-Einstein distribution $n(\varepsilon_k/T_s) = [\exp(\varepsilon_k/T_s) - 1]^{-1}$, where $\varepsilon_k = \Theta_C(ak)^2$ is the dispersion law for magnons in the long-wavelength limit $ka \ll 1$, with Θ_C being the Curie temperature, a the lattice constant, and $k = |\mathbf{k}|$ the magnon wave vector. The theoretical task is to derive *microscopically* the nonlinear heat current Q from hot magnons at temperature T_s to cold phonons at temperature T_l.

To accomplish this, we calculate the change in the number of phonons with the given wave vector $\mathbf{q}$ per unit of time $(\dot{N}_\mathbf{q})_s$ via the phonon-magnon collision integral $L_{ls}\{N, n\}$ [37] describing the absorption or emission of phonons by magnons, $(\dot{N}_\mathbf{q})_s = L_{ls}\{N, n\}$. Given the momentum conservation, $L_{ls}\{N, n\}$ can be expressed as

$$L_{ls}\{N, n\} = \frac{2\pi}{\hbar}\sum_{\mathbf{k}} |\psi_{sl}(\mathbf{q}, \mathbf{k}|\mathbf{k}+\mathbf{q})|^2\{(N_\mathbf{q}+1)(n_\mathbf{k}+1)n_{\mathbf{k}+\mathbf{q}} - N_\mathbf{q} n_\mathbf{k}(n_{\mathbf{k}+\mathbf{q}}+1)\} \times \delta(\hbar\omega_q + \varepsilon_\mathbf{k} - \varepsilon_{\mathbf{k}+\mathbf{q}}). \quad (1)$$

Here, $|\psi_{sl}(\mathbf{q}, \mathbf{k}|\mathbf{k}+\mathbf{q})|^2$ is the squared matrix element of the transition probability. It reads [37]

$$|\psi_{sl}(\mathbf{q}, \mathbf{k}|\mathbf{k}+\mathbf{q})|^2 = \frac{{\Theta_C}^2}{N}\left(\frac{\hbar}{\rho a^3\omega_q}\right)a^4k^2(\mathbf{k}+\mathbf{q})^2q^2, \quad (2)$$

where $\rho = M/a^3$, M is the mass of the magnetic ion, a is the lattice constant, Θ_C is the Curie temperature, N is the number of atoms, $\omega_q = sq$ is the frequency of phonons with the wavevector $\mathbf{q}$, s is the average speed of sound, and δ is the Dirac delta function.

In Eq. (1), $N_\mathbf{q}$ and $n_\mathbf{k}$ are the equilibrium Bose-Einstein distributions for phonons at temperature T_l and magnons at temperature T_s, namely,

$$N_\mathbf{q} = \frac{1}{\exp[(\hbar\omega_q/T_l) - 1]}, \quad n_\mathbf{k} = \frac{1}{\exp[(\varepsilon_k/T_s) - 1]}, \quad (3)$$

where $\varepsilon_k = \Theta_C(ak)^2$ is the dispersion law for magnons in the long-wavelength limit $ka \ll 1$. In the limiting case $T_l = T_s$, from Eq. (1) it follows that $L_{ls}\{N, n\} = 0$.

With the calculation steps detailed in the Appendix, the searched-for change in the number of phonons reads

$$\dot{N}_\mathbf{q} = D(T_s)[n(\varepsilon_q/T_s) - n(\varepsilon_q/T_l)]\sum_{p=1}^{\infty}(1 - e^{-px}) \times \int_{y_0}^{\infty} dy(yx + y^2)e^{-py}. \quad (4)$$

Here, $D(T_s) = (\Theta_C\Theta_D/8\pi\hbar\Theta_p)(T_s/\Theta_C)^3$, $\Theta_D = \hbar s/a$, $\Theta_p = Ms^2$, $x \equiv \varepsilon_q/T_l = \hbar\omega_q/T_l$, and $y_0 = \Theta_D^2/4T_s\Theta_C$, which plays the role of an effective inverse temperature. In the integral over the dimensionless magnon energy $y = \varepsilon_k/T_s$, the lower integration limit y_0 reflects the Cherenkov character

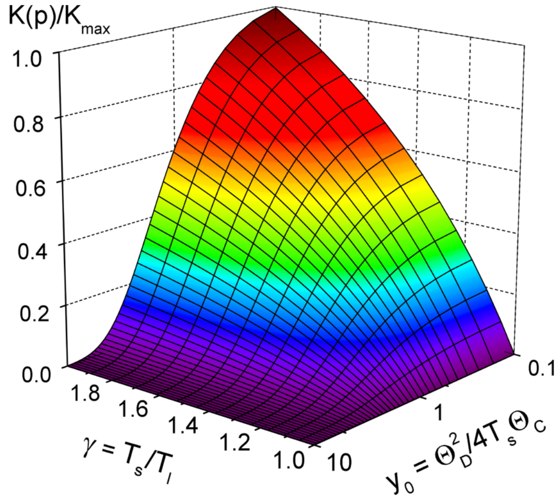

FIG. 2. The integral $K(p)$ calculated by Eq. (6) as a function of the magnon "overheating" parameter $\gamma = T_s/T_l$ and the effective inverse magnon temperature $y_0 = \Theta_D^2/4T_s\Theta_C$, normalized to its value $K_{\max}$ at $\gamma = 2$ and $y_0 = 0.1$.

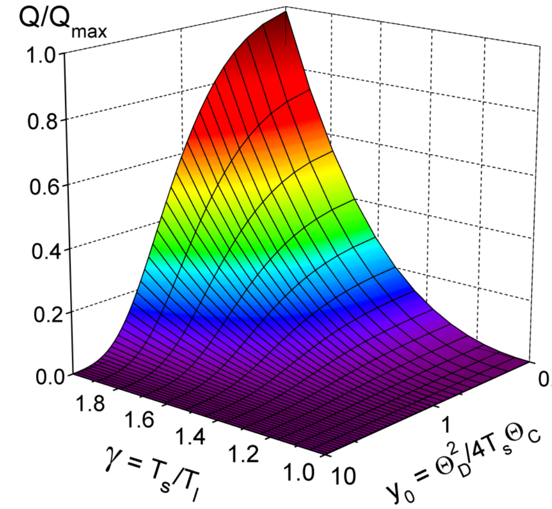

FIG. 3. The nonlinear heat current Q calculated by Eq. (5) as a function of the magnon "overheating" parameter $\gamma = T_s/T_l$ and the effective inverse magnon temperature $y_0 = \Theta_D^2/4T_s\Theta_C$, normalized to its value $Q_{\max}$ at $\gamma = 2$ and $y_0 = 0.1$.

of the emission of phonons by magnons. Namely, *only magnons whose energy is higher than* $\Theta_D^2/4\Theta_C$ *can emit phonons*.

With the passage from summation over **k** to integration and after the introduction of the magnon "overheating" parameter $\gamma = T_s/T_l$, the heat current $Q = \sum_{\mathbf{q}}(\hbar\omega_{\mathbf{q}})\dot{N}_{\mathbf{q}}$ from magnons to phonons acquires the form

$$Q = (N/8\pi^3)\big(\Theta_D^2\Theta_C/2\hbar\Theta_p\big)(T_s/\Theta_C)^3 \times [(T_s/\Theta_D)^4 - (T_l/\Theta_D)^4]K(p), \tag{5}$$

where

$$K(p) = \int_0^\infty \frac{u^3 du}{e^u - 1}[J_D(T_s, x = u, y_0) - J_D(T_s, x = u/\gamma, y_0)] \tag{6}$$

and

$$J_D(T) = \sum_{p=1}^{\infty}(1 - e^{-px})e^{-py_0}\left[x\left(\frac{y_0}{p} + \frac{1}{p^2}\right) + \left(\frac{y_0^2}{p} + \frac{2y_0}{p^2} + \frac{2}{p^3}\right)\right]. \tag{7}$$

The dependence of the integral $K(p)$ on the parameter $\gamma = T_s/T_l$ and the effective inverse magnon temperature $y_0 = \Theta_D^2/4T_s\Theta_C$ is illustrated in Fig. 2. One sees that when the magnon and phonon temperatures are equal, i.e., when $\gamma = 1$, $K(p) = 0$ as expected. In the limiting case of large y_0, which corresponds to the limit of low temperatures, $K(p)$ becomes exponentially small due to the factor $\sim e^{-y_0}$ in Eq. (7). The value of $K(p)$ increases with an increase in the magnon "overheating" parameter γ and a decrease in the inverse magnon temperature y_0. Accordingly, the nonlinear heat current Q, which is proportional to $K(p)$ and the factor $[\gamma^4 - 1]$, also increases with an increase in γ and a decrease in y_0, as illustrated in Fig. 3. Namely, Q increases with an increase in the difference between the magnon and the phonon temperatures, as well as with an increase in the magnon temperature T_s, as expected.

While Eqs. (5)–(7) are valid at any arbitrary temperature T_l when $T_s \ll \Theta_D$, the condition $T_s \ll \Theta_D$ allows us to essentially simplify Eq. (7) in the low-temperature limit. Namely, we can limit ourselves to $p = 1$ when $y_0(T_s) = \Theta_D^2/4T_s\Theta_C \gg 1$, since $J_D(T_s) \sim e^{-2y_0} \ll 1$ for $p = 2$. Namely, at $T_s \ll \Theta_D$

$$K(p = 1) = \varphi_1\Gamma(5)[1 + \mu[\zeta(5, 1 + \mu) - \zeta(5)]] + \varphi_2\Gamma(4)[1 + \mu[\zeta(4, 1 + \mu) - \zeta(4)]], \tag{8}$$

where $\Gamma(n)$ is the gamma function, $\zeta(n, m)$ is the generalized zeta function, $\mu = 1/\gamma = T_l/T_s$, $\varphi_1 = e^{-y_0}(y_0 + 1)$, and $\varphi_2 = e^{-y_0}(y_0^2 + 2y_0 + 2)$. The final result for $Q(p = 1)$ is obtained by substituting Eq. (8) into Eq. (5).

Finally, in addition to $T_s \ll \Theta_D$ we consider the limiting case $\gamma \to 1$ where the difference between the magnon and the phonon temperatures is small, that is, $(T_s - T_l) \ll T_s$. Accordingly, for $Q \equiv AB = A(T_s/\Theta_C)^3[(T_s/\Theta_D)^4 - (T_l/\Theta_D)^4]K(p = 1)$ with $A = (N/8\pi^3)(\Theta_D^2\Theta_C/2\hbar\Theta_p)$ we obtain $B \approx 4T_s^6(T_s - T_l)/(\Theta_C^3\Theta_D^4)$. In this way, the linear regime $Q \propto (T_s - T_l)$ is recovered from Eq. (5) in the limiting case $T_s \ll \Theta_D$ and $(T_s - T_l) \ll T_s$. This linear regime corresponds to $\gamma \to 1$ and $y_0 \to \infty$ and it is therefore beyond the y_0 range in Fig. 3.

III. DISCUSSION

Proceeding to a discussion of the obtained results, first we recall that Eqs. (5)–(7) describe the nonlinear heat current between *magnons and phonons* in an insulating ferromagnet in the case where the states of the magnon and phonon subsystems are described by the equilibrium Bose-Einstein distributions with different temperatures T_s and T_l, respectively. Experimentally, the condition $T_s > T_l$ can be realized as a consequence of, e.g., parametric pumping of spin waves in insulating ferromagnets [37]. Theoretically, the formulation of the considered problem is conceptually similar to the two-temperature KLT problem [15] of nonlinear relaxation between *electrons and phonons* in a metallic sample. Since the KLT model is widely used for analyzing experiments on the energy relaxation of excited electrons in metals [2,22–25], in what follows it is instructive to briefly outline the main results of the KLT work with the aid of emphasizing its similarities to and differences from the magnon-phonon nonlinear relaxation problem considered here.

Specifically, the KLT work relies upon a quadratic and isotropic dispersion of the electron energy in a metal $\epsilon_{\mathbf{p}} = p^2/2m$, where m is the effective mass. It is assumed that phonons have only a longitudinal acoustic mode with the linear dispersion $\omega_q = sq$, where s is the speed of longitudinal sound and $q = |\mathbf{q}|$ is the phonon wave vector. KLT use a deformation potential approximation for the electron-phonon interaction (EPI) [15]. Namely, the probability of the electron transition from the state with momentum $\mathbf{p}$ to the state with momentum $\mathbf{p}\prime$ per unit of time is expressed by the function $w(q)$, which is proportional to the squared EPI matrix element

$$w(q) = \frac{\pi \mu^2 \omega_q}{\rho_f s^2}, \tag{9}$$

where μ is the constant of the deformation potential of the order of the Fermi energy $\mu \sim \varepsilon_F = p_F^2/2m$ and ρ_f is the film density. In the KLT work, electrons and phonons are considered at quasiequilibrium and they are characterized by the temperatures T_e and T_p, respectively.

For derivation of the dynamic equations for the electron and phonon temperatures, KLT derived the specific power P_{ep} of the heat current from hot electrons to cold phonons, which is expressed via the electron-phonon collision integral

$$P_{ep} = \int \frac{d^3 q}{(2\pi)^3}\, \hbar\omega_q\, I_{pe}(N_{\mathbf{q}}, f_{\mathbf{p}}). \tag{10}$$

With the Bose-Einstein distribution $N_{\mathbf{q}} = n_q \equiv [\exp(\hbar\omega_q/k_B T_p) - 1]^{-1}$ for phonons and the Fermi distribution $f_{\mathbf{p}} = f_0(\epsilon_p) \equiv \{\exp[(\epsilon_p - \epsilon_F)/k_B T_e] + 1\}^{-1}$ for electrons, KLT obtained the following expression for P_{ep}, which is valid at arbitrary temperatures [15]:

$$P_{ep}(T_e, T_p) = \frac{m^2 \mu^2 (k_B \Theta_D)^5}{4\pi^3 \hbar^7 \rho_f s^4}[F(T_e) - F(T_p)], \tag{11}$$

where the function $F(T)$ is determined by

$$F(T) = \left(\frac{T}{\Theta_D}\right)^5 \int_0^{\Theta_D/T} \frac{x^4\, dx}{e^x - 1}. \tag{12}$$

From Eqs. (11) and (12) it follows that at high temperatures (with respect to Θ_D) $P_{ep} = \alpha(T_e - T_p)$, while at low temperatures $P_{ep} = A(T_e^5 - T_p^5)$. The constants $\alpha = (m^2 \mu^2 k_B^5 \Theta_D^4)/(16\pi^3 \hbar^7 \rho_f s^4)$ and $A = (D_5 m^2 \mu^2 k_B^5)/(4\pi^3 \hbar^7 \rho_f s^4)$ do not depend on the electron and phonon temperatures and determine the strength of the EPI at high and low temperatures, respectively. In the last equality, $D_5 \approx 24.9$ is the integration result of $D_k = \int_0^\infty x^{k-1}(e^x - 1)^{-1}\, dx$ at $k = 5$. On the basis of the KLT work [15] one can write the system of the nonlinear dynamic equations for the electron and phonon temperatures [21,22]. In the spatially homogeneous case, which is typical for thin films, this system of equations reads

$$c_e(T_e)\frac{d\,T_e}{d\,t} = -P_{ep}(T_e, T_p) + W(t), \tag{13}$$

$$c_p(T_p)\frac{d\,T_p}{d\,t} = P_{ep}(T_e, T_p), \tag{14}$$

where c_e and c_p are the electron and magnon specific heats, respectively, and $W(t)$ is the specific power of heat sources heating the electrons.

Turning back to our magnon-phonon problem, in the spatially homogeneous case of an insulating ferromagnetic thin film with $d < s/\nu_{ls}$, where d is the film thickness and ν_{ls} is the collision frequency of phonons with magnons, we can write a system of the nonlinear dynamic equations for the magnon and phonon temperatures,

$$c_s(T_s)\frac{d\,T_s}{d\,t} = -Q(T_s, T_l) + W_s(t), \tag{15}$$

$$c_l(T_l)\frac{d\,T_l}{d\,t} = Q(T_s, T_l), \tag{16}$$

where c_s and c_l are the magnon and phonon specific heats, respectively, and $W_s(t)$ is the specific power of heat sources heating the magnons.

Now, we are in a position to emphasize the similarities and the differences in the results obtained on the problems of nonlinear magnon-phonon relaxation, in our work, and nonlinear electron-phonon relaxation, in the KLT work. First, the general scheme for calculation of the heat flows in both problems is formally similar, relying upon the formulas

$$Q = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} \dot{N}_{\mathbf{q}}(T_s, T_l), \tag{17}$$

$$P_{ep} = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} \dot{N}_{\mathbf{q}}(T_e, T_p), \tag{18}$$

where $\dot{N}_{\mathbf{q}}$ is the change in the number of phonons with the wave vector $\mathbf{q}$ per unit of time. This change in the number of phonons is caused by the emission or absorption of phonons by magnons [Eq. (17)] or electrons [Eq. (18)], and it is determined by collision integrals (1) and (10) for phonons with the respective quasiparticles. Both these collision integrals are equal to the product of the frequency ν of the collisions of phonons with magnons or electrons and the difference in the equilibrium Bose-Einstein distributions $n(\varepsilon_q/T)$, namely,

$$L_{ls} = \nu_{ls}[n(\varepsilon_q/T_s) - n(\varepsilon_q/T_l)], \tag{19}$$

$$I_{pe} = \nu_{pe}[n(\varepsilon_q/T_e) - n(\varepsilon_q/T_p)], \tag{20}$$

where $\varepsilon_q = \hbar\omega_q$ is the phonon energy.

Second, we note that while the integrals L_{ls} and I_{pe} in Eqs. (19) and (20) look formally similar, the collision integral for magnons and phonons L_{ls} given by Eq. (1) for the collision frequency ν_{ls} has a more complex structure than that for the collision frequency of phonons with electrons $\nu_{pe} \sim (s/\vartheta_F)\omega_q$ given by Eq. (10).

Third, the presence of the finite integration limit y_0 over the dimensionless magnon energy $y = \varepsilon_k/T$ in Eq. (4) is caused by the fact that the emission of phonons by magnons is only possible for magnons whose energy is higher than $\Theta_D^2/4\Theta_C$. It is this crucial point which underlines the *Cherenkov character* of emission of phonons by magnons in insulating ferromagnets. This is distinct from the EPI in metals, where any electron at the Fermi surface can absorb and emit a phonon, since the speed of sound in metals s is much smaller than the electron Fermi velocity ϑ_F. Consequently, in contrast to the frequency of the phonon-electron collision ν_{pe}, which only depends on the absolute value of the phonon wave vector q, the frequency of the phonon-magnon collisions in Eq. (4) also *depends on the magnon temperature* T_s, that is,

$$\nu_{ls}(T_s, q) = D(T_s)J_D(T_s). \tag{21}$$

In addition, we note that the expression for Q in Eq. (5) is only valid when $T_s \ll \Theta_D$, while for electrons in metals the expression $P_{ep}(T_e, T_p)$ is valid at *any arbitrary* T_e or T_p when $T_e \ll \varepsilon_F$. The same considerations hold for the nonlinear dynamic equations for electrons [Eqs. (13) and (14)] and magnons [Eqs. (15) and (16)].

Finally, we would like to emphasize the general importance of the obtained results. In the experimental work by Schreier *et al.* [2] it has been pointed out that one of the challenges in analyzing intertwinned charge, spin, and heat currents in hybrid magnetic structures is a proper account of temperature differences in the electron, magnon, and phonon subsystems, caused by the different thermal properties and boundary conditions for the respective quasiparticles. The phonon, electron, and magnon temperature profiles in substrate/ferromagnet/normal metal multilayers can exhibit discontinuities at the material interfaces due to interface properties such as the Kapitza resistance [36]. The temperature profiles are not easily measurable for a nonequilibrium situation in which magnon, phonon, and electron temperatures differ. In-depth analysis and interpretation of experimental spin Seebeck effect data are, to date, possible only by modeling the magnon, phonon, and electron temperature profiles based on the relevant material parameters [2]. Especially for magnetic insulators determination of the phonon temperature profile is of central importance. In particular, linear-response theories for the heat flow across an interface, relying upon the Kapitza thermal boundary resistance [36], can be used for interpreting experimental data acquired at room temperature [2,39,40]. By contrast, in experiments at low temperatures [29–31] and/or at high-microwave-power spin pumping [32], the magnon temperature can be essentially higher than the phonon temperature, thus necessitating the consideration of the nonlinear heat current regime. At the same time, the experimental condition of low temperatures, at which the nonlinearity of the Kapitza resistance is most pronounced, justifies our assumption of the magnon temperature being much lower than the Debye temperature, $T_s \ll \Theta_D$. Accordingly, the elaborated theoretical account of the nonlinear heat current from hot magnons to cold phonons in insulating ferromagnets sets the foundation for a followup calculation of the *nonlinear* thermal boundary resistance in multilayer spin caloritronic structures and analysis of the magnon and phonon temperature profiles in the nonlinear low-temperature regime.

IV. CONCLUSION

To conclude, we have theoretically investigated the nonlinear relaxation between magnons and phonons in an insulating ferromagnet. Magnons and phonons were described by equilibrium Bose-Einstein distributions with different temperatures. The nonlinear heat current from magnons to phonons has been calculated microscopically in terms of the Cherenkov radiation of phonons by magnons. The main messages and results of this work can be summarized as follows: (i) For a large difference between the magnon and the phonon temperatures the *nonlinear* magnon-phonon relaxation is important. (ii) The nonlinear relaxation regime is realized experimentally at low temperatures. (iii) The condition of low temperatures has allowed for treating the nonlinear magnon-phonon relaxation under the assumption that the magnon temperature is much lower than the Debye temperature. (iv) The nonlinear heat current by Eq. (5) can be used for calculation of the heat current [41] and the nonlinear heat boundary resistance in spin-caloritronic heterostructures at low temperatures. In all, the elaborated theoretical account is relevant for low-temperature spin-pumping experiments and thermoelectric devices in which the magnon temperature is essentially higher than the phonon one.

ACKNOWLEDGMENT

Research leading to these results received funding from the European Commission in the framework of the program Marie Sklodowska-Curie Actions—Research and Innovation Staff Exchange (MSCA-RISE) under Grant Agreement No. 644348 (MagIC) call: H2020-MSCA-RISE-2014.

APPENDIX

This Appendix addresses the calculation of the collision integral given by Eq. (1). To this end, the curly bracket in Eq. (1) is denoted Φ and the new variables $x \equiv \varepsilon_q/T_l = \hbar\omega_q/T_l$ and $y \equiv \varepsilon_k/T_s$ are introduced. Then Φ acquires the form

$$\Phi = \left(\frac{1}{e^x - 1} - \frac{1}{e^{x\gamma} - 1}\right)\left[\frac{e^y}{e^y - 1} - \frac{e^{y+x}}{e^{y+x} - 1}\right], \tag{A1}$$

where $\gamma = T_s/T_l > 1$. Here, we have used the relations

$$\frac{1}{(e^{y+x} - 1)(e^y - 1)} = \frac{1}{e^x - 1}\left[\frac{1}{e^y - 1} - \frac{e^x}{e^{y+x} - 1}\right],$$

$$\frac{e^{\gamma x} - e^x}{(e^{\gamma x} - 1)(e^x - 1)} = \frac{1}{e^x - 1} - \frac{1}{e^{\gamma x - 1}}.$$

Condition (A1) for Φ can be rewritten in terms of the sum of the geometric sequences with decreasing denominators e^{-y} and $e^{-(x+y)}$, namely,

$$\Phi = [n(\varepsilon_k/T_s) - n(\varepsilon_q/T_l)]\sum_{p=1}^{\infty} e^{-py}(1 - e^{-px}).$$

When passing from the sum over $\mathbf{k}$ to integration in Eq. (A1) in the long-wavelength limit $ka \ll 1$, we have used

$$\sum_{\mathbf{k}} \rightarrow \frac{V}{(2\pi)^3}\int d\mathbf{k} = \frac{Na^3}{(2\pi)^3}\int k^2 dk dO,$$

where $dO = 2\pi \sin\theta d\theta$ and θ is the polar angle of the vector $\mathbf{k}$ with respect to the vector $\mathbf{q}$. Given that

$$\delta(\hbar\omega_q + \varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{q+k}}) = \frac{\delta(f - \cos\theta)}{\Theta_C(2a^2qk)},$$

where $f = (1/2ak)((\Theta_D/\Theta_C) - qa)$, one obtains Eq. (4):

$$\dot{N}_{\mathbf{q}} = D(T_s)[n(\varepsilon_q/T_s) - n(\varepsilon_q/T_l)]\sum_{p=1}^{\infty}(1 - e^{-px}) \times \int_{y_0}^{\infty} dy(yx + y^2)e^{-py},$$

where $D(T) = (\Theta_C\Theta_D/8\pi\hbar\Theta_p)(T/\Theta_C)^3$, $y_0 = \Theta_D^2/4T_s\Theta_C$, $\Theta_D = \hbar s/a$, and $\Theta_p = Ms^2$. For the calculation of

$$J_D(T) = \sum_{p=1}^{\infty}(1 - e^{-px})\int_{y_0}^{\infty} dy(yx + y)e^{-py}$$

one rewrites it as

$$J_D(T) = \sum_{p=1}^{\infty}(1 - e^{-px})e^{-py_0}\left[x\left(\frac{y_0}{p} + \frac{1}{p^2}\right) + \left(\frac{y_0^2}{p} + \frac{2y_0}{p^2} + \frac{2}{p^3}\right)\right]. \quad \text{(A2)}$$

Noting that $J_D(T_s) \sim e^{-2y_0} \ll 1$ for $p = 2$ since $y_0(T_s) = \Theta_D^2/4T_s\Theta_C \gg 1$ we can limit ourselves by $p = 1$, obtaining

$$J_D(T_s, p = 1) \approx (1 - e^{-x})e^{-y_0}\left[x(y_0 + 1) + y_0^2 + 2y_0 + 2\right].$$

The heat current from magnons to phonons is determined by

$$Q = \textstyle\sum_{\mathbf{q}}(\hbar\omega_{\mathbf{q}})\dot{N}_{\mathbf{q}} = \sum_{\mathbf{q}}(\hbar\omega_q)D(T_s)J_D(T_s, x, y_0)[n(\varepsilon_q/T_s) - n(\varepsilon_q/T_l)],$$

where $J_D(T_s, x, y_0)$ is given by Eq. (23). By passing from $\sum_{\mathbf{q}}$ to the integral one can show that

$$Q = (N/8\pi^3)\left(\Theta_D^2\Theta_C/2\hbar\Theta_p\right)(T_s/\Theta_C)^3 \times [(T_s/\Theta_D)^4 - (T_l/\Theta_D)^4] \times \int_0^{\infty}\left(\frac{u^3du}{e^u - 1}\right)[J_D(T_s, x = u, y_0) - J_D(T_s, x = u/\gamma, y_0)]. \quad \text{(A3)}$$

Here, the calculation of Q at an arbitrary p is reduced to the calculation of the integral

$$K(p) = \int_0^{\infty}\frac{u^3du}{e^u - 1}[J_D(T_s, x = u, y_0) - J_D(T_s, x = u/\gamma, y_0)].$$

Using relation 2.3.13.22 in Ref. [42] for $p = 1$ one can rewrite

$$\int_0^{\infty}(u^{n-1}e^{-u}du)/(e^u - 1) = \Gamma(n)[\zeta(n, 2)],$$

where $\Gamma(n)$ is the gamma function and $\zeta(n, 2)$ is the generalized zeta function. Then

$$K(p = 1) = \varphi_1\Gamma(5)[1 + \mu[\zeta(5, 1 + \mu) - \zeta(5)]] + \varphi_2\Gamma(4)[1 + \mu[\zeta(4, 1 + \mu) - \zeta(4)]]. \quad \text{(A4)}$$

Here, $\varphi_1 = e^{-y_0}(y_0 + 1)$, $\varphi_2 = e^{-y_0}(y_0^2 + 2y_0 + 2)$, and $\mu = 1/\gamma = T_l/T_s$. The final result for $Q(p = 1)$ is obtained by the substitution of Eq. (A4) into Eq. (A3).

[1] G. E. W. Bauer, E. Saitoh, and B. J. van Wees, Nat. Mater. **11**, 391 EP (2012).
[2] M. Schreier, A. Kamra, M. Weiler, J. Xiao, G. E. W. Bauer, R. Gross, and S. T. B. Goennenwein, Phys. Rev. B **88**, 094410 (2013).
[3] S. R. Boona, R. C. Myers, and J. P. Heremans, Energy Environ. Sci. **7**, 885 (2014).
[4] K. Uchida, S. Takahashi, K. Harii, J. Ieda, W. Koshibae, K. Ando, S. Maekawa, and E. Saitoh, Nature **455**, 778 EP (2008).
[5] M. Weiler, M. Althammer, M. Schreier, J. Lotze, M. Pernpeintner, S. Meyer, H. Huebl, R. Gross, A. Kamra, J. Xiao, Y.-T. Chen, H. J. Jiao, G. E. W. Bauer, and S. T. B. Goennenwein, Phys. Rev. Lett. **111**, 176601 (2013).
[6] T. B. Noack, H. Y. Musiienko-Shmarova, T. Langner, F. Heussner, V. Lauer, B. Heinz, B. D. A., V. I. Vasyuchka, A. Pomyalov, V. S. Lvov, B. Hillebrands, and A. A. Serga, J. Phys. D: Appl. Phys. **51**, 234003 (2018).
[7] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, and B. Hillebrands, Nat. Phys. **11**, 453 (2015).
[8] A. V. Chumak, A. A. Serga, and B. Hillebrands, Nat. Commun. **5**, 4700 (2014).
[9] D. Grundler, Nat. Nanotech. **11**, 407 (2016).
[10] J. Cramer, F. Fuhrmann, U. Ritzmann, V. Gall, T. Niizeki, R. Ramos, Z. Qiu, D. Hou, T. Kikkawa, J. Sinova, U. Nowak, E. Saitoh, and M. Kläui, Nat. Commun. **9**, 1089 (2018).
[11] D. J. Sanders and D. Walton, Phys. Rev. B **15**, 1489 (1977).
[12] T. An, V. I. Vasyuchka, K. Uchida, A. V. Chumak, K. Yamaguchi, K. Harii, J. Ohe, M. B. Jungfleisch, Y. Kajiwara, H. Adachi, B. Hillebrands, S. Maekawa, and E. Saitoh, Nat. Mater. **12**, 549 (2013).
[13] A. A. Serga, A. V. Chumak, and B. Hillebrands, J. Phys. D: Appl. Phys. **43**, 264002 (2010).
[14] C. M. Bhandari and G. S. Verma, Phys. Rev. **152**, 731 (1966).
[15] M. Kaganov, I. M. Lifshitz, and L. V. Tanatarov, J. Exp. Theor. Phys. **31**, 232 (1956).

[16] V. V. Kabanov and A. S. Alexandrov, Phys. Rev. B **78**, 174514 (2008).
[17] J. Xiao, G. E. W. Bauer, K.-c. Uchida, E. Saitoh, and S. Maekawa, Phys. Rev. B **81**, 214418 (2010).
[18] A. I. Bezuglyj and V. A. Shklovskij, Low Temp. Phys. **42**, 636 (2016).
[19] Y. Liu, L.-S. Xie, Z. Yuan, and K. Xia, Phys. Rev. B **96**, 174416 (2017).
[20] A. I. Bezuglyj and V. A. Shklovskij, J. Phys.: Condens. Matter **30**, 295001 (2018).
[21] S. I. Anisimov, Y. A. Imas, G. S. Romanov, and Y. V. Khodyko, *Action of High-Power Radiation on Metals* (Moscow, Nauka, 1970).
[22] S. I. Anisimov, B. L. Kapeliovich, and T. L. Perelman, Sov. Phys. JETP **39**, 375 (1974).
[23] J. G. Fujimoto, J. M. Liu, E. P. Ippen, and N. Bloembergen, Phys. Rev. Lett. **53**, 1837 (1984).
[24] H. E. Elsayed-Ali, T. B. Norris, M. A. Pessot, and G. A. Mourou, Phys. Rev. Lett. **58**, 1212 (1987).
[25] R. W. Schoenlein, W. Z. Lin, J. G. Fujimoto, and G. L. Eesley, Phys. Rev. Lett. **58**, 1680 (1987).
[26] S. Hoffman, K. Sato, and Y. Tserkovnyak, Phys. Rev. B **88**, 064408 (2013).
[27] K. Miyazaki and K. Seki, J. Chem. Phys. **108**, 7052 (1998).
[28] A. Akhieser, J. Phys. USSR **10**, 247 (1946).
[29] B. L. Giles, Z. Yang, J. S. Jamison, and R. C. Myers, Phys. Rev. B **92**, 224415 (2015).
[30] E.-J. Guo, J. Cramer, A. Kehlberger, C. A. Ferguson, D. A. MacLaren, G. Jakob, and M. Kläui, Phys. Rev. X **6**, 031012 (2016).
[31] A. Prakash, B. Flebus, J. Brangham, F. Yang, Y. Tserkovnyak, and J. P. Heremans, Phys. Rev. B **97**, 020408 (2018).
[32] M. B. Jungfleisch, A. V. Chumak, A. Kehlberger, V. Lauer, D. H. Kim, M. C. Onbasli, C. A. Ross, M. Kläui, and B. Hillebrands, Phys. Rev. B **91**, 134407 (2015).
[33] S. A. Wolf, D. D. Awschalom, R. A. Buhrman, J. M. Daughton, S. von Molnár, M. L. Roukes, A. Y. Chtchelkanova, and D. M. Treger, Science **294**, 1488 (2001).
[34] I. Zutic, J. Fabian, and S. Das Sarma, Rev. Mod. Phys. **76**, 323 (2004).
[35] J. Sinova and I. Zutic, Nat. Mater. **11**, 368 (2012).
[36] P. L. Kapitza, J. Phys. **4**, 181 (1941).
[37] A. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminski, *Spin Waves* (Moscow, Nauka, 1967).
[38] J. Kimling, G.-M. Choi, J. T. Brangham, T. Matalla-Wagner, T. Huebner, T. Kuschel, F. Yang, and D. G. Cahill, Phys. Rev. Lett. **118**, 057201 (2017).
[39] L. J. Cornelissen, J. Liu, R. A. Duine, J. B. Youssef, and B. J. van Wees, Nat. Phys. **11**, 1022 (2015).
[40] L. J. Cornelissen, K. J. H. Peters, G. E. W. Bauer, R. A. Duine, and B. J. van Wees, Phys. Rev. B **94**, 014412 (2016).
[41] V. A. Shklovskij, V. V. Kruglyak, R. V. Vovk, and O. V. Dobrovolskiy, https://arxiv.org/abs/1808.07294.
[42] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series. Elementary Functions* (Moscow, Nauka, 2002).